

CONSTRUCTING APPROXIMATE FIBRATIONS

BY

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ABSTRACT. In this paper two results concerning the construction of approximate fibrations are established. The first shows that there are approximate fibrations $p: M \rightarrow S^2$ which are homotopic to bundle maps but which cannot be approximated by bundle maps. Here M can be a compact Q -manifold or some topological n -manifold, $n \geq 5$. The second shows how to construct approximate fibrations $p: M \rightarrow B$ whose fibers do not have finite homotopy type, for any B of Euler characteristic zero. Here M can be a compact Q -manifold and B only has to be an ANR, or M can be an n -manifold, $n \geq 6$, and B must then also be a topological manifold.

1. Introduction. Let E and B be locally compact separable metric ANRs and let $p: E \rightarrow B$ be a *proper map*, i.e., the preimages of compacta are compact. We say that p is an *approximate fibration* provided that given any X and maps $F: X \times [0, 1] \rightarrow B$, $f: X \rightarrow E$ for which $F(x, 0) = pf(x)$, then for every $\varepsilon > 0$ there exists a map $\tilde{F}: X \times [0, 1] \rightarrow E$ such that $\tilde{F}(x, 0) = f(x)$ and $p\tilde{F}(x, t)$ is ε -close to $F(x, t)$, for all $(x, t) \in X \times [0, 1]$. This concept was introduced by Coram and Duvall in [10] as a generalization of the notion of a Hurewicz fibration. It has been studied extensively by several authors because approximate fibrations seem to arise more naturally than do Hurewicz fibrations or bundle maps when one studies certain topological manifold problems.

In this paper we will only be interested in approximate fibrations $p: E \rightarrow B$ for which $E = M^k$, $k \leq \infty$, where M^k is a closed topological k -manifold for $k < \infty$, and is a compact Hilbert cube manifold (i.e., Q -manifold) for $k = \infty$. (Recall that a *Q -manifold* is a separable metric manifold modeled on the Hilbert cube Q .) Our main results are Theorems 1 and 2 where we show how to construct examples of approximate fibrations which are homotopic to bundle maps but which cannot be approximated by bundle maps, and also examples of approximate fibrations with nonfinite fibers.

In [17] Husch showed that an approximate fibration $p: M^k \rightarrow S^1$ can be approximated arbitrarily closely by a bundle map iff it is homotopic to a bundle map, $6 \leq k \leq \infty$. By a bundle map we mean a locally trivial bundle map with fiber an ANR. The following theorem shows that some restrictions are necessary in order to extend this result to more complicated bases. It also gives a negative answer to question QM15 of [15].

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THEOREM 1. *For any k , $5 \leq k \leq \infty$, there exists an approximate fibration $p: M^k \rightarrow S^2$ which is homotopic to a bundle map but which cannot be approximated arbitrarily closely by bundle maps (or even Hurewicz fibrations).*

REMARKS. (1) In the Q -manifold case the fiber of $p: M^\infty \rightarrow S^2$ can be taken to be any finite complex F for which $K_{-1}Z\pi_1(F) \neq 0$, where K_{-1} is the lower algebraic K -theoretic functor of Bass [2]. Calculations of Carter assure us that $K_{-1}Z\pi_1(F) \neq 0$, for $\pi_1(F) \cong Z_2 \times Z_7$ [3].

(2) It is also true that if $K_{1-n}Z\pi_1(F) \neq 0$, then the proof of Theorem 1 gives us an approximate fibration $p: M^\infty \rightarrow S^n$ with fiber F which is homotopic to a bundle map but which cannot be approximated arbitrarily closely by bundle maps.

(3) In the finite-dimensional case the fiber of $p: M^k \rightarrow S^2$ can be taken to be any closed manifold F^{k-2} for which the duality involution, $\bar{}: K_{-1}Z\pi_1(F) \rightarrow K_{-1}Z\pi_1(F)$, which is induced by the involution $x \rightarrow x^{-1}$ on $\pi_1(F)$, does not act trivially or by multiplication by -1 . Fortunately for $n = 2$ the example $\pi_1(F) = Z_2 \times Z_7$ of Carter satisfies this property [3].

(4) Also for the finite-dimensional case if F^{k-n} is chosen so that $\bar{}: K_{1-n}Z\pi_1(F) \rightarrow K_{1-n}Z\pi_1(F)$ does not act trivially or by multiplication by -1 , then the proof of Theorem 1 gives us an approximate fibration $p: M^k \rightarrow S^n$ with fiber F which is homotopic to a bundle map but which cannot be approximated arbitrarily closely by one.

(5) Certainly it would be nice to have techniques and calculations at our disposal which would extend Theorem 1 to include bases which are more complicated than spheres.

In order to state our next result we will have to recall some facts about approximate fibrations. If $p: E \rightarrow B$ is an approximate fibration of compact ANRs and B is connected, then the homotopy fiber of p is shape equivalent to all of the point-inverses of p . Also the homotopy fiber of p is *finitely dominated*, i.e., is homotopy dominated by a finite complex. These facts follow easily from Proposition 2.3 of [6], or the reader can construct his own proofs. So it becomes a problem to determine which finitely dominated complexes can be realized as the homotopy fibers of approximate fibrations $p: M^k \rightarrow B$. In [12] Ferry showed that if F is any finitely dominated complex, then there is an approximate fibration $p: M^\infty \rightarrow S^1$ with homotopy fiber F . In [14] Ferry constructed an example of an approximate fibration $p: M^k \rightarrow S^1$, $6 \leq k < \infty$, whose homotopy fiber is nonfinite, i.e., does not have the homotopy type of a finite complex. The question was then raised of the possibility of obtaining approximate fibrations with nonfinite fibers over bases more complicated than S^1 [15, QM13].

THEOREM 2. *For any k , $6 \leq k \leq \infty$, there exists an approximate fibration $p: M^k \rightarrow B$ with nonfinite fiber, where*

- (1) *B is any compact connected ANR having Euler characteristic zero, if $k = \infty$,*
- (2) *B is a closed connected n -manifold, $k - n \geq 4$, having Euler characteristic zero, if $k < \infty$.*

REMARKS. (1) In the infinite-dimensional case the proof of Theorem 2 actually yields the following stronger result: *If B is a compact connected ANR and F is a finitely dominated complex with Wall finiteness obstruction $\sigma(F) \in \tilde{K}_0 Z\pi_1(F)$ which satisfies $\chi(B) \cdot \sigma(F) = 0$ (χ = Euler characteristic), then F is the homotopy fiber of some approximate fibration $p: M^\infty \rightarrow B$.*

(2) In the finite-dimensional case the proof of Theorem 2 actually yields the following stronger result: *If B^n is a closed connected manifold and F is a finitely dominated complex for which $\sigma(F) = \mu + (-1)^{k-n}\bar{\mu}$, for some $\mu \in \tilde{K}_0 Z\pi_1(F)$ and where $\bar{\cdot}: \tilde{K}_0 Z\pi_1(F) \rightarrow \tilde{K}_0 Z\pi_1(F)$ is the duality involution induced by the involution $x \rightarrow x^{-1}$ on $\pi_1(F)$, then there is an approximate fibration $p: M^k \rightarrow B^n$ whose homotopy fiber has finiteness obstruction equal to $\sigma(F)$ provided that $\chi(B) \cdot \sigma(F) = 0$.*

(3) A related problem is to determine just what finiteness obstructions are realizable in fibers of approximate fibrations of the above type. Specifically we ask: *If B is a compact connected ANR and G is a finitely presented group, what elements of $\tilde{K}_0 ZG$ can be realized as the finiteness obstruction of the homotopy fiber F of some approximate fibration $p: M^\infty \rightarrow B$, where $\pi_1(F) = G$? It is not hard to show that for any such $p: M^\infty \rightarrow B$ we must have $i_*\chi(B) \cdot \sigma(F) = 0$, where i is the inclusion $F \hookrightarrow M$ and $i_*: \tilde{K}_0 Z\pi_1(F) \rightarrow \tilde{K}_0 Z\pi_1(M)$ is induced by $\pi_1(F) \rightarrow \pi_1(M)$. Thus if $\pi_1(F) \rightarrow \pi_1(M)$ has a left inverse, then it follows that $\chi(B) \cdot \sigma(F) = 0$ and so by Remark (1) above we have a complete determination of all possible $\sigma(F)$'s in this case. Even if $\pi_1(F) \rightarrow \pi_1(M)$ does not have a left inverse then it is still possible to conjecture that $\chi(B) \cdot \sigma(F)$ must be zero. For example one can show that if $p: M^\infty \rightarrow S^2$ is an approximate fibration with fiber F , then $2\sigma(F) = 0$. The general case seems difficult.*

(4) As in Remark (3) above there is the realization problem for finite-dimensional manifolds. Again it seems difficult.

2. Preliminaries. All spaces in this paper will be locally compact, separable and metric. If $h_t: X \rightarrow Y$ is a homotopy, $0 \leq t \leq 1$, and $p: Y \rightarrow B$ is a map, then h_t is said to be a $p^{-1}(\epsilon)$ -homotopy provided that each set $\{ph_t(x) \mid 0 \leq t \leq 1\}$ has diameter $< \epsilon$. A proper map $f: X \rightarrow Y$ is said to be a $p^{-1}(\epsilon)$ -equivalence provided that there is a proper map $g: Y \rightarrow X$ so that gf is proper $(pf)^{-1}(\epsilon)$ -homotopic to id_X and fg is proper $p^{-1}(\epsilon)$ -homotopic to id_Y . If $Y = B$ and $p = \text{id}$, then f is simply called an ϵ -equivalence.

For every pair of maps $f, g: X \rightarrow Y$ and $A \subset Y$ we say that $f = g$ over A means that $f^{-1}(A) = g^{-1}(A)$ and the restrictions $f|_{f^{-1}(A)}, g|_{g^{-1}(A)}$ are equal. For a given base B and maps $f: X \rightarrow Y, p: X \rightarrow B$ and $q: Y \rightarrow B$, we say that f is *fiber preserving* (f.p.) if $qf = p$.

If $f: X \rightarrow Y$ is a map of compacta, then we define the *mapping cylinder* of f, M_f , to be $(X \times [0, 1]) \sqcup Y/\sim$, where $\sqcup$ means disjoint union and $\sim$ is the equivalence relation generated by $(x, 1) \sim f(x)$. We will represent M_f as the union of $X \times [0, 1]$ and Y , where Y is the *base* and $X \equiv X \times \{0\}$ is the *top*. The *collapse to the base* is the retraction $c: M_f \rightarrow Y$ defined by $c|_Y = \text{id}$ and $c(x, t) = f(x)$.

Numerous concepts and results from Q -manifold theory will be used in the sequel such as Z -sets, Z -set unknotting, the Triangulation Theorem, and the Classification

Theorem which relates the study of homeomorphisms on Q -manifolds to simple homotopy theory. We refer the reader to [8] for the appropriate Q -manifold theory.

A closed subset A of a Q -manifold M is said to be *clean* if A is a Q -manifold and its topological boundary, $\text{Bd}(A)$, is also a Q -manifold which is collared in A and also in $M - \mathring{A}$ ($\mathring{A}$ = topological interior of A). Similarly a closed subset A of a PL manifold M is said to be *clean* if A is a PL submanifold which satisfies the above conditions and which lies in the interior of M .

We use R^n to denote Euclidean n -space, with $R^1 = R$, and we use $T^n = S^1 \times \cdots \times S^1$ to denote the n -torus. For any $r \geq 0$ we define $B_r^n = [-r, r]^n$.

Recall from [9] that to each homotopy equivalence $f: X \rightarrow Y$ of compact polyhedra (or compact ANRs [8]) we can assign a torsion $\tau(f) \in \text{Wh}(Y)$ which vanishes iff f is a simple homotopy equivalence. We will use several standard facts from simple homotopy theory such as the Sum Theorem, the formula for the torsion of a composition, etc. An excellent reference for all of this is [9].

We indicated in §1 that for each finitely dominated polyhedron X we can assign an element $\sigma(X) \in \tilde{K}_0 Z\pi_1(X)$ which vanishes iff X is homotopy equivalent to a compact polyhedron. For simplicity we write $\tilde{K}_0 Z\pi_1(X) = \tilde{K}_0(X)$ and by direct summing over path components we can extend the definitions to include nonconnected spaces. Thus $\tilde{K}_0$ (as well as Wh) becomes a homotopy functor from the category of spaces and maps to the category of abelian groups and homomorphisms. We will need the Sum Theorem for the Wall obstruction $\sigma(X)$. A good reference for this is [19, Chapter VI].

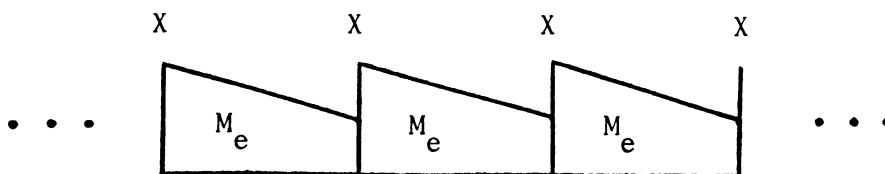
3. Proof of Theorem 1 (infinite-dimensional case). Before getting into the proof of Theorem 1 it will be convenient to establish the following lemma. Choose a point $x_0 \in S^2$ and identify R^2 with $S^2 - \{x_0\}$. In this section we will alternately use π to denote projection to T^2 , R^2 and S^2 . The meaning should be clear from context.

LEMMA 3.1. *For every $\epsilon > 0$ there exists a $\delta > 0$ so that if $f: M \rightarrow R^2 \times F$ is a $\pi^{-1}(\delta)$ -equivalence, where M is a Q -manifold and F is a compact Q -manifold, then there exists a compact Q -manifold N and a $\pi^{-1}(\epsilon)$ -equivalence $g: N \rightarrow S^2 \times F$ for which $g = f$ over $B_1^2 \times F$.*

PROOF. We will use mapping cylinder tricks that are similar to those used in [13]. Without loss of generality we may assume that f is an inclusion and M is a Z -set in $R^2 \times F$. Also we may assume that for certain specific values of t which arise in the following proof, $M \cap (B_t^2 \times F)$ is clean in M . In particular it follows that the set $X = M \cup [(S^2 - \mathring{B}_3^2) \times F]$ is a compact Q -manifold because it is the union of the compact Q -manifolds $M \cap (B_3^2 \times F)$ and $(S^2 - \mathring{B}_3^2) \times F$ which meet in the Q -manifold $M \cap (\partial B_3^2 \times F)$ which is a Z -set in each. Since f is a $\pi^{-1}(\delta)$ -equivalence it follows that $X \hookrightarrow S^2 \times F$ is a $\pi^{-1}(\delta')$ -domination rel M , i.e., there is a $\pi^{-1}(\delta')$ -homotopy $h_i: S^2 \times F \rightarrow S^2 \times F$ for which $h_0 = \text{id}$, $h_i|_M = \text{id}$, and $h_1(S^2 \times F) \subset X$. (The homotopy h_i comes from [4] by using a controlled version of the homotopy extension theorem and the size of δ' depends on the size of δ .) Let $e = h_1|_X: X \rightarrow X$ and form the mapping telescope, S_e , which is the quotient space obtained from the disjoint union

$$\cdots \sqcup (X \times [-1, 0]) \sqcup (X \times [0, 1]) \sqcup (X \times [1, 2]) \sqcup \cdots$$

by identifying (x, n) in $X \times [n-1, n]$ with $(e(x), n)$ in $X \times [n, n+1]$. Note that S_e is just the union of a countable number of copies of the mapping cylinder M_e . Here is a picture:



In a natural way S_e may be set-wise identified with $X \times R$ (noncontinuously).

By the proof of Theorem 3.1 of [13] there is a homotopy equivalence $H: S_e \rightarrow S^2 \times F$ for which $H(x, 0) = x$, for $x \in X$, and $H(x, t) = x$, for all (x, t) in $M \times R$. This is a (nonproper) $\pi^{-1}(\delta'')$ -equivalence, where the size of δ'' depends on the size of δ' . If $i: X \rightarrow S_e$ is defined by $i(x) = (x, 0)$, then we will construct N and a nonproper $(\pi H)^{-1}(\epsilon')$ -equivalence $k: N \rightarrow S_e$ for which $k = i$ over $H^{-1}(B_1^2 \times F)$, where the size of ϵ' depends on the size of δ'' . Once this is done we can define our desired $g: N \rightarrow S^2 \times F$ by $g = HK$.

Consider the decomposition $X = X_1 \cup X_2$ where $X_1 = X - (\mathring{B}_{2,5}^2 \times F)$ and $X_2 = X \cap (\mathring{B}_{2,5}^2 \times F)$. By our assumption that $M \cap (\mathring{B}_{2,5}^2 \times F)$ is clean in M we conclude that X_1 , X_2 and $X_0 = X_1 \cap X_2$ are all Q -manifolds. For δ sufficiently small we have a decomposition $S_e = S_{e_1} \cup (X_2 \times R)$, where $e_1 = e|_{X_1}: X_1 \rightarrow X_1$. The homotopy equivalence H transfers the homotopy domination $X \rightarrow S^2 \times F \text{ rel } M$ to give a homotopy domination $X \rightarrow S_e \text{ rel } M$. (We are identifying X with $i(X)$ in S_e .) Using this it is easy to see that (for δ small) $X_1 \rightarrow S_{e_1}$ is also a homotopy domination, thus the Wall obstruction $\sigma(S_{e_1}) \in \tilde{K}_0(S_{e_1})$ is defined. We have

$$\sigma(S_e) = \sigma(X_0 \times R) = \sigma(X_2 \times R) = 0,$$

so by the Sum Theorem for the Wall obstruction we conclude that $j_*\sigma(S_{e_1}) = 0$, where $j_*: \tilde{K}_0(S_{e_1}) \rightarrow \tilde{K}_0(S_e)$ is induced by $j: S_{e_1} \hookrightarrow S_e$.

If δ is small enough the homotopy equivalence H , along with the functorality of $\tilde{K}_0$, provides us with a homomorphism

$$\phi: \tilde{K}_0(S_e) \rightarrow \tilde{K}_0(S_{e_1} \cup (X_2 - (\mathring{B}_{2,4}^2 \times F)) \times R)$$

so that $l_* = \phi j_*$, where l is the inclusion $S_{e_1} \hookrightarrow S_{e_1} \cup (X_2 - (\mathring{B}_{2,4}^2 \times F)) \times R$. From this we conclude that $l_*\sigma(S_{e_1}) = 0$, and therefore

$$\sigma(S_{e_1} \cup (X_2 - (\mathring{B}_{2,4}^2 \times F)) \times R) = 0.$$

This means that the map $i: X \rightarrow S_e$ can be extended to our desired homotopy equivalence $k: N \rightarrow S_e$ by adding a compact polyhedron or, equivalently, a compact Q -manifold to $X - (\mathring{B}_{2,4}^2 \times F)$ to make

$$X - (\mathring{B}_{2,4}^2 \times F) \hookrightarrow S_{e_1} \cup (X_2 - (\mathring{B}_{2,4}^2 \times F)) \times R$$

into a homotopy equivalence. It is easy to get k to be a $(\pi H)^{-1}(\epsilon')$ -equivalence simply by choosing the identification $R^2 \equiv S^2 - \{x_0\}$ so that the diameter of $S^2 - \mathring{B}_2^2$ is small. $\square$

PROOF OF THEOREM 1 (INFINITE-DIMENSIONAL CASE). We start by choosing any compact connected Q -manifold F for which $K_{-1}Z\pi_1(F) \neq 0$. Since $K_{-1}Z\pi_1(F)$ is a direct summand of $\text{Wh}(T^2 \times F) = \text{Wh}Z\pi_1(T^2 \times F)$ [1] we can choose a compact Q -manifold M_1 and a homotopy equivalence $f_1: M_1 \rightarrow T^2 \times F$ whose Whitehead torsion, $\tau(f_1)$, is a nonzero element of the subgroup $K_{-1}Z\pi_1(F)$ of $\text{Wh}(T^2 \times F)$. Such torsions are invariant under passage to standard finite covers, so we may assume that f_1 is a $\pi^{-1}(\varepsilon_1)$ -equivalence, where $\pi = \text{proj}: T^2 \times F \rightarrow T^2$ and ε_1 is a small number (compare with Lemma 7.3 of [6]).

Let $e: R \rightarrow T^1$ be the covering map defined by $e(x) = e^{\pi i x}$, thus giving the covering map $e^n = e \times \cdots \times e: R^n \rightarrow T^n$. Form the pull-back,

$$\begin{array}{ccc} M_2 & \xrightarrow{f_2} & R^2 \times F \\ \downarrow & & \downarrow e^2 \times \text{id} \\ M_1 & \xrightarrow{f_1} & T^2 \times F, \end{array}$$

where $M_2 \subset M_1 \times (R^2 \times F)$ consists of those elements (x, y) of $M_1 \times (R^2 \times F)$ for which $f_1(x) = (e^2 \times \text{id})(y)$. The map f_2 is a projection map and it is a $\pi^{-1}(\varepsilon_2)$ -equivalence, where ε_2 depends on ε_1 . Also the map $M_2 \rightarrow M_1$ is an immersion, and therefore M_2 is a Q -manifold. These are standard facts about the pull-back construction which have appeared several times in the literature (see, for example, the proof of Theorem 2 of [6]). Using Lemma 3.1 we can find a $\pi^{-1}(\varepsilon_3)$ -equivalence $f_3: M_3 \rightarrow S^2 \times F$ where M_3 is a compact Q -manifold, $f_3 = f_2$ over $B_1^2 \times F$, and ε_3 depends on ε_2 . For k large and ε_3 small we can choose a clean manifold $C \subset M_3$ which lies in $f_3^{-1}((S^2 - \dot{B}_k^2) \times F)$ for which the inclusion-induced homomorphism $\text{Wh}(C) \rightarrow \text{Wh}(M_3)$ is surjective.

We can find a compact Q -manifold $\tilde{C}$ containing C as a Z -set so that there exists a strong deformation retraction $r: \tilde{C} \rightarrow C$ whose torsion, $\tau(r) \in \text{Wh}(C)$, is sent to $-\tau(f_3) \in \text{Wh}(S^2 \times F)$ under the composition

$$\text{Wh}(C) \rightarrow \text{Wh}(M_3) \xrightarrow{(f_3)^*} \text{Wh}(S^2 \times F).$$

Let $M = M_3 \cup \tilde{C}$ (sewn together along C), which is still a Q -manifold, and let $f: M \rightarrow S^2 \times F$ be the homotopy equivalence obtained by first retracting $\tilde{C}$ to C via r , and then applying f_3 . Observe that f is a $\pi^{-1}(\varepsilon)$ -equivalence, where ε is small if k is large and ε_3 is small. Also we have $\tau(f) = 0$ by the formula for the torsion of a composition [9, p. 72], so f is homotopic to a homeomorphism $h: M \rightarrow S^2 \times F$ by the Classification Theorem of [8]. In the language of [6] the map $\pi f: M \rightarrow S^2$ is ε' -fibration, where ε' is small if ε is small. Thus by Theorem 1 of [6] we can choose ε' small enough so that πf is close to an approximate fibration $p: M \rightarrow S^2$. We have a homotopy $p \simeq \pi f \simeq \pi h$, the latter map being a bundle map. So we will be done when we have shown that p is not close to a bundle map.

That p is not close to a bundle map is a consequence of the proof of Theorem 2 of [6]. We outline the proof. Embed $S^1 \times R^1$ in S^2 . Restricting p to the inverse image of $S^1 \times R^1$ gives an approximate fibration over $S^1 \times R^1$. Controlled engulfing and

Theorem 1 of [6] produce an approximate fibration over T^2 and a controlled homotopy equivalence from the total space of this approximate fibration to $T^2 \times F$. The torsion of this homotopy equivalence lies in $\text{Wh}(T^2 \times F)$. A major portion of the proof of Theorem 2 of [6] consists of showing that the $\tilde{K}_{-1}$ component of the torsion is independent of the original embedding and of the wrapping up process.

It is now clear that p is not approximated by a bundle map. If this were possible, then one could easily perform the embedding and wrapping up process so that the $\tilde{K}_{-1}$ component of the resulting torsion would be zero. On the other hand, the process can clearly be performed in such a way as to recover the original manifold M_1 and homotopy equivalence f_1 , so the $\tilde{K}_{-1}$ component of the torsion is nonzero.

4. Proof of Theorem 1 (finite-dimensional case). In this section we will be dealing with PL h -cobordisms (W, N) and it will always be understood that $\partial W = \partial_0 W \cup N$, where $\partial_0 W \cap N = \emptyset$. Also we will always have a given proper map π of N to some base space B , and by a $\pi^{-1}(\epsilon)$ -sdr (sdr $\equiv$ strong deformation retraction) we will mean a proper retraction $f: W \rightarrow N$ which is $(\pi f)^{-1}(\epsilon)$ -homotopic to $\text{id}_W \text{ rel } N$. As in our treatment of the infinite-dimensional case in §3 it will be convenient to first establish an extension lemma. Our basic notation is the same as in §3: π denotes projection to T^2 , R^2 or S^2 , and we assume the identification $R^2 \equiv S^2 - \{x_0\}$.

LEMMA 4.1. *For every $\epsilon > 0$ there exists a $\delta > 0$ so that if $(W^{k+1}, R^2 \times F)$ is an h -cobordism ($k \geq 5$), $f: W \rightarrow R^2 \times F$ is a $\pi^{-1}(\delta)$ -sdr, and $\partial_0 W \hookrightarrow W$ is a $(\pi f)^{-1}(\delta)$ -equivalence, then there exists an h -cobordism $(V, S^2 \times F)$ and a $\pi^{-1}(\epsilon)$ -sdr $g: V \rightarrow S^2 \times F$ for which $\partial_0 V \hookrightarrow V$ is a $(\pi g)^{-1}(\epsilon)$ -equivalence and $g = f$ over $B_1^2 \times F$.*

PROOF. Using the geometrical connectivity result of [18, p. 308] we may assume that W is a handlebody structure on $R^2 \times F$ consisting of two layers of handles of indices $k-2$ and $k-1$, where the image of each handle under the map $\pi f: W \rightarrow R^2$ has a small diameter. Let W' be the union of $S^2 \times F$ and all handles of W which meet $(\pi f)^{-1}(B_3^2)$. Then W' is a handlebody structure on $S^2 \times F$ and the restriction $f|: W' \rightarrow S^2 \times F$ is a $\pi^{-1}(\delta)$ -equivalence over $B_2^2 \times F$. It then follows from the trick on p. 324 of [18] that we can alter this handlebody structure over $(S^2 - \dot{B}_2^2) \times F$ to form a new h -cobordism $(V, S^2 \times F)$, and if $S^2 - \dot{B}_2^2$ has a sufficiently small diameter there is a $\pi^{-1}(\epsilon)$ -sdr $g: V \rightarrow S^2 \times F$ which fulfills our requirements. $\square$

PROOF OF THEOREM 1 (FINITE-DIMENSIONAL CASE). We follow fairly closely the proof of the infinite-dimensional case of §3. Choose a compact connected manifold F^{k-2} for which the duality involution $\bar{\cdot}: K_{-1}Z\pi_1(F) \rightarrow K_{-1}Z\pi_1(F)$ does not act trivially or by multiplication by -1 . As indicated in §1 all we need to do is choose $\pi_1(F) = Z_2 \times Z_7$ to fulfill these requirements. Thus we can choose an element $\tau \in K_{-1}Z\pi_1(F) \subset \text{Wh}(T^2 \times F)$ for which

$$\bar{\tau} \neq \begin{cases} \tau & \text{if } k \text{ is even,} \\ -\tau & \text{if } k \text{ is odd.} \end{cases}$$

Now form an h -cobordism $(W_1, T^2 \times F)$ such that there exists a sdr $f_1: W_1 \rightarrow T^2 \times F$ which has torsion τ . This of course means that $\partial_0 W_1 \hookrightarrow W_1$ is also a homotopy equivalence. Here $\partial_0 W_1$ denotes the other boundary component of W_1 . If $u: T^2 \rightarrow T^2$

is any standard finite cover we can form the pull-back

$$\begin{array}{ccc} W_2 & \xrightarrow{f_2} & T^2 \times F \\ \downarrow & & \downarrow u \times \text{id} \\ W_1 & \xrightarrow{f_1} & T^2 \times F. \end{array}$$

If u is a large enough cover we conclude that f_2 is a $\pi^{-1}(\varepsilon_2)$ -sdr, where ε_2 is a small number. By the same token it follows that we can also choose u large enough so that $\partial_0 W_2 \hookrightarrow W_2$ is a $(\pi f_2)^{-1}(\varepsilon_2)$ -equivalence.

The next step is to form the pull-back,

$$\begin{array}{ccc} W_3 & \xrightarrow{f_3} & R^2 \times F \\ \downarrow & & \downarrow e^2 \times \text{id} \\ W_2 & \xrightarrow{f_2} & T^2 \times F. \end{array}$$

Then $(W_3, R^2 \times F)$ is an h -cobordism, f_3 is a $\pi^{-1}(\varepsilon_3)$ -sdr, and $\partial_0 W_3 \hookrightarrow W_3$ is a $(\pi f_3)^{-1}(\varepsilon_3)$ -equivalence, where ε_3 depends on ε_2 . Using Lemma 4.1 we can find an h -cobordism $(W_4, S^2 \times F)$ and a $\pi^{-1}(\varepsilon_4)$ -sdr $f_4: W_4 \rightarrow S^2 \times F$ for which $\partial_0 W_4 \hookrightarrow W_4$ is a $(\pi f_4)^{-1}(\varepsilon_4)$ -equivalence and $f_4 = f_3$ over $B_1^2 \times F$, where ε_4 depends on ε_3 . If r is large and ε_4 is sufficiently small we can find a clean PL submanifold C^k of $\partial_0 W_4$ which lies in $f_4^{-1}((S^2 - \tilde{B}_r^2) \times F)$ and for which the inclusion-induced homomorphism $\text{Wh}(C) \rightarrow \text{Wh}(W_4)$ is surjective. Let $(\tilde{C}, C)$ be an h -cobordism rel ∂C for which the torsion of the sdr $s: \tilde{C} \rightarrow C$ is sent to $-\tau(f_4)$ under the composition

$$\text{Wh}(C) \rightarrow \text{Wh}(W_4) \xrightarrow{(f_4)^*} \text{Wh}(S^2 \times F).$$

Let $W = W_4 \cup \tilde{C}$ (sewn together along C), which defines a new h -cobordism $(W, S^2 \times F)$. The sdr $f: W \rightarrow S^2 \times F$, which is defined by first retracting $\tilde{C}$ to C and then applying f_4 , is a $\pi^{-1}(\varepsilon)$ -sdr for ε small provided that r is large and ε_4 is small. Also we have $\tau(f) = 0$, so by the s -cobordism theorem the cobordism $(W, S^2 \times F)$ is trivial, i.e., $(W, S^2 \times F)$ is homeomorphic to $(S^2 \times F \times [0, 1], S^2 \times F)$.

Let $M = \partial_0 W$ and note that, in the language of [7], the composition $M \xrightarrow{i} W \xrightarrow{\pi f} S^2$ is an ε' -fibration, where ε' is small if ε is small. By Theorem 1 of [7] we conclude that $\pi f i: M \rightarrow S^2$ is close to an approximate fibration $p: M \rightarrow S^2$. Since the cobordism $(W, S^2 \times F)$ is trivial we conclude that p is homotopic to a bundle map. We will be done when we have shown that p is not close to a bundle map. To do this we can multiply everything by Q to obtain Q -manifolds, and then repeat the proof of the corresponding Q -manifold assertion of §3. To carry this out we need to know that the torsion of the composition $\partial_0 W_1 \xrightarrow{j} W_1 \xrightarrow{f_1} T^2 \times F$ is nonzero. Recall from [16, p. 273] that we have the formula $\tau(j) = (-1)^k \overline{t(I)}$, where $l: T^2 \times F \hookrightarrow W_1$ and $\bar{\cdot}: \text{Wh}(W_1) \rightarrow \text{Wh}(W_1)$ is the duality involution that is compatible with the one on

$K_{-1}Z\pi_1(F)$ that has already been mentioned. From this it is easy to calculate $\tau(f_1j) = \tau + (-1)^{k+1}\bar{\tau}$, which is nonzero by our choice of τ . $\square$

5. Controlled finiteness theorems. The purpose of this section is to establish two results (Theorems 5.2 and 5.3) which will be needed in the proof of Theorem 2. Specifically we will examine the following question: *If B is a compact connected polyhedron and F is a finitely dominated polyhedron, when is there a compact polyhedron K and a $p^{-1}(\epsilon)$ -equivalence $K \rightarrow B \times F$, where $p = \text{proj}: B \times F \rightarrow B$ and ϵ is small?* In Theorem 5.2 we show that this is true iff $\chi(B) \cdot \sigma(F) = 0$, where $\sigma(F) \in \tilde{K}_0(F)$ is the Wall obstruction of F and $\chi(B)$ is the Euler characteristic of B . Theorem 5.3 is merely a refinement of Theorem 5.2 with dimension estimates.

Before we establish these results we will first have to prove a lemma. For notation let B be a compact connected polyhedron and let $p: E \rightarrow B$ be a PL map which is an approximate fibration, where E is also a compact polyhedron. For a fixed basepoint $* \in B$ we assume that there is a retraction $r: E \rightarrow p^{-1}(*)$. Now let $p_1: E_1 \rightarrow B$ be another such PL approximate fibration so that E_1 contains E as a subpolyhedron and so that there is a f.p. retraction $f_1: E_1 \rightarrow E$ which is f.p. homotopic to $\text{id rel } E$. This can be rephrased by saying that f_1 is a f.p. sdr. For each simplex $\Delta \subset B$ (in a fixed triangulation of B) the homotopy equivalence $f_1|: p_1^{-1}(\Delta) \rightarrow p^{-1}(\Delta)$ gives us a torsion $\tau(f_1|) \in \text{Wh}(p^{-1}(\Delta))$ and we let τ_Δ be the image of $\tau(f_1|)$ in $\text{Wh}(p^{-1}(*))$ under the composition

$$\text{Wh}(p^{-1}(\Delta)) \rightarrow \text{Wh}(E) \xrightarrow{r^*} \text{Wh}(p^{-1}(*)).$$

If B_1 is any subcomplex of B , then we inductively define $\tau_{B_1} \in \text{Wh}(p^{-1}(*))$ by $\tau_{B_1} = \tau_{B_2} + \tau_\Delta - \tau_{\partial\Delta}$, where $B_1 = B_2 \cup \Delta$ and $\Delta \cap B_2 = \partial\Delta$. It is easy to show that if $\tau(f_1) \in \text{Wh}(E)$ is the usual torsion of f_1 , then $\tau(f_1)$ is the image of τ_B under the composition $\text{Wh}(p^{-1}(*)) \rightarrow \text{Wh}(E)$.

LEMMA 5.1. *If $\tau_B = 0$, then for every $\epsilon > 0$ there exists a homeomorphism $h: E_1 \times Q \rightarrow E \times Q$ which is $(p \circ \text{proj})^{-1}(\epsilon)$ -homotopic to $f_1 \times \text{id}_Q$.*

PROOF. There is a trick of trading adjacent cancelling torsions which will be useful. To explain this we look at the simplified case $B = [0, 1]$ (a single 1-simplex) and show how torsions can be traded to concentrate τ_B over $*$, which we take to be $1 \in [0, 1]$. In what follows all torsions are in $\text{Wh}(p^{-1}(*))$. Choose a fine subdivision of $[0, 1]$, $0 = t_0 < t_1 < \dots < t_{2k+1} = 1$, and form $E_2 = E_1 \cup (\bigcup_{i=0}^{2k+1} A_i)$, where the A_i are disjoint compact polyhedra so that A_i meets E_1 in the subpolyhedron $p^{-1}(t_i)$. Also we want there to be a PL sdr $r_i: A_i \rightarrow p^{-1}(t_i)$ so that

- (1) $\tau(r_{2i}) + \tau(f_1|_{p_1^{-1}(t_{2i})}) = 0$, $0 \leq i \leq k$,
- (2) $\tau(r_{2i+1}) + \tau(r_{2i}) = 0$, $0 \leq i \leq k$.

The sdr's r_i piece together to give us a PL sdr $f_2: E_2 \rightarrow E_1$ so that $p_2 = p_1 f_2: E_2 \rightarrow B$ is a PL approximate fibration. By (2) we have formed E_2 by adding torsions to E_1 in adjacent cancelling pairs. It follows from the Classification Theorem of [8], along

²The reader should note that this is an isomorphism.

with Z -set unknotting, that $f_2 \times \text{id}_Q: E_2 \times Q \rightarrow E_1 \times Q$ is $(p_1 \circ \text{proj})^{-1}(\epsilon)$ -homotopic to a homeomorphism, where the size of ϵ is determined by the mesh of the partition $\{t_i\}$ of $[0, 1]$. This homeomorphism is constructed by piecing together $k + 1$ homeomorphisms with disjoint supports, each support containing a cancelling pair.

For the remainder of this case we examine $f_1 f_2: E_2 \rightarrow E$. The situation is now somewhat improved because the torsion of each $f_1 f_2|: p_2^{-1}(t_{2i}) \rightarrow p^{-1}(t_{2i})$ is 0 by (1) above. The next step is to trade off the torsions

$$\begin{aligned}\tau_{2i} &= f_1 f_2|: p_2^{-1}([t_{2i}, t_{2i+2}]) \rightarrow p^{-1}([t_{2i}, t_{2i+2}]), \quad 0 \leq i \leq k-1, \\ \tau_{2k} &= f_1 f_2|: p_2^{-1}([t_{2k}, 1]) \rightarrow p^{-1}([t_{2k}, 1]).\end{aligned}$$

Observe that by the Sum Theorem we have $\tau_0 + \tau_2 + \cdots + \tau_{2k} = \tau_B$. Now choose an additional subdivision $c_{2i} < t_{2i} < d_{2i}$, $1 \leq i \leq k$, where c_{2i} and d_{2i} are close to t_{2i} . Just as E_2 was formed from E_1 we now form

$$E_3 = E_2 \cup \left(\bigcup_{i=1}^k C_i \right) \cup \left(\bigcup_{i=1}^k D_i \right).$$

It is easy to choose PL sdr's $C_i \rightarrow p_2^{-1}(c_{2i})$ and $D_i \rightarrow p_2^{-1}(d_{2i})$ which piece together to give a f.p. sdr $f_3: E_3 \rightarrow E_2$ for which

- (1) $\tau(f_3|C_i) + (\tau_0 + \cdots + \tau_{2i-2}) = 0$, $1 \leq i \leq k$,
- (2) $\tau(f_3|D_i) - (\tau_0 + \cdots + \tau_{2i-2}) = 0$, $1 \leq i \leq k$.

If we define $p_3 = p_2 f_3: E_3 \rightarrow B$, then p_3 is a PL approximate fibration. Since we have added torsions to E_2 in adjacent cancelling pairs we conclude that $f_3 \times \text{id}: E_3 \times Q \rightarrow E_2 \times Q$ is $(p_2 \circ \text{proj})^{-1}(\epsilon)$ -homotopic to a homeomorphism, where ϵ can be made as small as we want by choosing the differences $d_{2i} - c_{2i}$ small. For the remainder of this case we only have to consider $f_1 f_2 f_3: E_3 \rightarrow E$. We still have the torsion of each $f_1 f_2 f_3|: p_3^{-1}(t_{2i}) \rightarrow p^{-1}(t_{2i})$ zero. But, moreover, we now have the torsion of

$$f_1 f_2 f_3|: p_3^{-1}([t_{2i}, t_{2i+2}]) \rightarrow p^{-1}([t_{2i}, t_{2i+2}])$$

zero, and the torsion of

$$f_1 f_2 f_3|: p_3^{-1}([t_{2k}, 1]) \rightarrow p^{-1}([t_{2k}, 1])$$

is $\tau_0 + \cdots + \tau_{2k}$. So what we have done so far is to show how to concentrate τ_B over $[t_{2k}, 1]$. It is clear that by applying the same ideas again we can construct a PL approximate fibration $p_4: E_4 \rightarrow B$ and a f.p. PL sdr $f_4: E_4 \rightarrow E_3$ so that:

- (1) $f_4 \times \text{id}: E_4 \times Q \rightarrow E_3 \times Q$ is $(p \circ \text{proj})^{-1}(\epsilon)$ -homotopic to a homeomorphism, where the size of ϵ depends on $1 - t_{2k}$;
- (2) the torsion of $f_1 f_2 f_3 f_4|: p_4^{-1}(t_{2i}) \rightarrow p^{-1}(t_{2i})$ is 0, for all i ;
- (3) the torsion of $f_1 f_2 f_3 f_4|: p_4^{-1}([t_{2i}, t_{2i+2}]) \rightarrow p^{-1}([t_{2i}, t_{2i+2}])$ is 0, for all i ;
- (4) the torsion of $f_1 f_2 f_3 f_4|: p_4^{-1}([t_{2k}, 1]) \rightarrow p^{-1}([t_{2k}, 1])$ equals the torsion of $f_1 f_2 f_3 f_4|: p_4^{-1}(1) \rightarrow p^{-1}(1)$.

Thus the torsion of $f_1 f_2 f_3 f_4: E_4 \rightarrow E$ is concentrated over $*$.

We now use the above ideas to finish the proof of the lemma. For arbitrary B we can trade torsions to concentrate τ_B over $*$. By this we mean that there is a PL approximate fibration $p': E' \rightarrow B$ and a f.p. PL sdr $f': E' \rightarrow E_1$ so that:

- (1) $f' \times \text{id}: E' \times Q \rightarrow E_1 \times Q$ is $(p_1 \circ \text{proj})^{-1}(\epsilon)$ -homotopic to a homeomorphism, where ϵ is small;

(2) $\tau((p')^{-1}(\Delta) \xrightarrow{f_1 f'} p^{-1}(\Delta))$ is 0, where Δ is any simplex in $B^{(r)}$ which does not contain $*$ ($B^{(r)} = r$ th barycentric subdivision of B , r large);

(3)

$$\tau\left((p')^{-1}(\Delta) \xrightarrow{f_1 f'} p^{-1}(\Delta)\right) = \tau\left((p')^{-1}(*) \xrightarrow{f_1 f'} p^{-1}(*)\right) = \tau_B,$$

where Δ is any simplex in $B^{(r)}$ which does contain $*$.

Since we are given $\tau_B = 0$ we have $\tau((p')^{-1}(\Delta) \rightarrow p^{-1}(\Delta)) = 0$, for all simplices Δ in $B^{(r)}$. We will use this to show that $f_1 f' \times \text{id}: E' \times Q \rightarrow E \times Q$ is homotopic to a homeomorphism, where the size of the homotopy in B is determined by the diameters of the simplices in $B^{(r)}$. If r is large enough this homotopy must then be small enough to fulfill our requirements.

By the Classification Theorem of [8] it follows that each

$$f_1 f' \times \text{id}| : (p')^{-1}(\Delta) \times Q \rightarrow p^{-1}(\Delta) \times Q$$

is homotopic to a homeomorphism. If it is true that each $(p')^{-1}(\partial\Delta)$ is a Z -set in $(p')^{-1}(\Delta)$ and each $p^{-1}(\partial\Delta)$ is a Z -set in $p^{-1}(\Delta)$, then it is easy to use the Classification Theorem and Z -set unknotting to build a homotopy of $f_1 f' \times \text{id}$ to a homeomorphism which is *block-preserving*, i.e., the restriction of this homotopy to each $(p')^{-1}(\Delta) \times Q$ has image in $p^{-1}(\Delta) \times Q$. Since the Δ 's are simplices of $B^{(r)}$ we are done. If the Z -set property is not satisfied, then we can construct new PL approximate fibrations $\tilde{p}': \tilde{E}' \rightarrow B$, $\tilde{p}: \tilde{E} \rightarrow B$ and CE maps $\alpha: \tilde{E}' \rightarrow E'$, $\beta: \tilde{E} \rightarrow E$ so that

- (1) α and β are block-preserving,
- (2) each $(\tilde{p}')^{-1}(\partial\Delta)$ is a Z -set in $(\tilde{p}')^{-1}(\Delta)$,
- (3) each $(\tilde{p})^{-1}(\partial\Delta)$ is a Z -set in $(\tilde{p})^{-1}(\Delta)$.

(To see how to define $\tilde{E}$ for $\dim B = 1$ we first let $\tilde{p}^{-1}(v) = p^{-1}(v)$, for all vertices v . Then for Δ a 1-simplex we define

$$\tilde{p}^{-1}(\Delta) = p^{-1}(\Delta) \cup (p^{-1}(\partial\Delta) \times [0, 1])$$

(union along $p^{-1}(\partial\Delta) \equiv p^{-1}(\partial\Delta) \times \{0\}$), where $\tilde{p}^{-1}(\Delta)$ is sewn to $\tilde{p}^{-1}(\partial\Delta)$ along $p^{-1}(\partial\Delta) \times \{1\}$. In general the construction is inductive.) Now α and β stabilize to near homeomorphisms [8, p. 105], and the block-preserving property (1) assures us that we have appropriate small control in B . So all we need is a homeomorphism $h: \tilde{E}' \times Q \rightarrow \tilde{E} \times Q$ for which the following diagram $(p \circ \text{proj})^{-1}(\epsilon)$ -homotopy commutes:

$$\begin{array}{ccc} \tilde{E}' \times Q & \xrightarrow{h} & \tilde{E} \times Q \\ \alpha \times \text{id} \downarrow & & \downarrow \beta \times \text{id} \\ E' \times Q & \xrightarrow{f_1 f' \times \text{id}} & E \times Q \end{array}$$

This is easily obtained by first noting that there is a map $H: \tilde{E}' \rightarrow \tilde{E}$ which is block-preserving and for which βH is block-preserving homotopic to $f_1 f' \alpha$. Then

$H| : (\tilde{p}')^{-1}(\Delta) \rightarrow (\tilde{p})^{-1}(\Delta)$ has zero torsion, for all simplices Δ of $B^{(r)}$, and we can now use Z -set unknotting to obtain a homotopy of $H \times \text{id}_Q$ to a homeomorphism. $\square$

We are now ready for our main result. In the following statement B will be a compact connected polyhedron and p will denote projection to B .

THEOREM 5.2. *If F is a finitely dominated polyhedron, then the condition $\chi(B) \cdot \sigma(F) = 0$ is necessary and sufficient that for every $\epsilon > 0$ there exists a compact polyhedron K and a $p^{-1}(\epsilon)$ -equivalence $K \rightarrow B \times F$.*

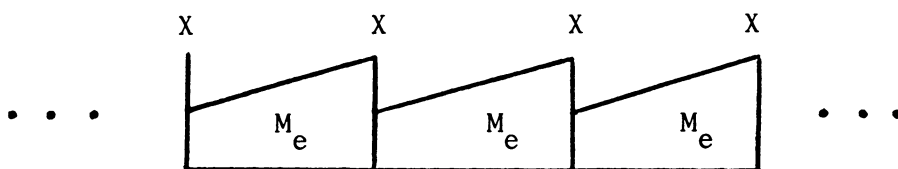
PROOF. If such a K exists, then we have $\sigma(B \times F) = 0$. By an induction on $\dim B$ it is easy to use the Sum Theorem for the Wall obstruction to prove that $\sigma(B \times F) = i_* \chi(B) \cdot \sigma(F)$, where $i: F \rightarrow B \times F$ is defined by $i(x) = (*, x)$. But i has a left inverse, so by the functoriality of σ we have $\chi(B) \cdot \sigma(F) = 0$.

For the other half of the result we assume that $\chi(B) \cdot \sigma(F) = 0$. Since F is finitely dominated there is a compact polyhedron X and maps $d: X \rightarrow F$, $u: F \rightarrow X$ for which $du: F \rightarrow F$ is homotopic to id_F . Let $e = ud: X \rightarrow X$ and form the mapping telescope, S_e , which was introduced in the proof of Lemma 3.1. We also need the *mapping torus*, $T_e = M_e / \sim$, where $\sim$ is the equivalence relation on the mapping cylinder M_e which is generated by $(x, 0) \sim e(x)$. If $\exp: R \rightarrow S^1$ is defined by $\exp(x) = e^{\pi i x}$, then it follows from the proof of Theorem 3.1 of [13] that there are homotopy equivalences $H: T_e \rightarrow F \times S^1$ and $\tilde{H}: S_e \rightarrow F \times R$ for which the following diagram commutes:

$$\begin{array}{ccc} S_e & \xrightarrow{\tilde{H}} & F \times R \\ \downarrow & & \downarrow \text{id} \times \exp \\ T_e & \xrightarrow{H} & F \times S^1 \end{array}$$

The map $S_e \rightarrow T_e$ is the obvious covering map obtained by identifying each (x, t) in S_e with the equivalence class of (x, t) in $M_e / \sim$.

By reversing the ends of R we similarly obtain the *reverse mapping telescope*, S_e^- , which is pictured as follows:



Note that rays of each mapping cylinder go from right to left. Similarly we obtain the *reverse mapping torus*, T_e^- , and we obtain homotopy equivalences $H_-: T_e^- \rightarrow F \times S^1$, $\tilde{H}_-: S_e^- \rightarrow F \times R$ for which the following diagram commutes:

$$\begin{array}{ccc} S_e^- & \xrightarrow{\tilde{H}_-} & F \times R \\ \downarrow & & \downarrow \text{id} \times \exp \\ T_e^- & \xrightarrow{H_-} & F \times S^1 \end{array}$$

In the natural set-wise identification $S_e \equiv X \times R$ we observe that each $X \times [n, n+1]$ is a copy of M_e with $X \times \{n+1\}$ being the base. In the natural set-wise identification $S_e^- \equiv X \times R$ we have each $X \times [n, n+1]$ also a copy of M_e , but $X \times \{n\}$ is now the base. The maps $\tilde{H}$ and $\tilde{H}_-$ are actually proper homotopy equivalences which preserve the induced orientations. We can choose a homotopy inverse of H_- , $H_-^{-1}: F \times S^1 \rightarrow T_e^-$, and a proper homotopy inverse of $\tilde{H}_-$, $\tilde{H}_-^{-1}: F \times R \rightarrow S_e^-$, so that $\tilde{H}_-^{-1}$ covers H_-^{-1} .

Armed with the above notation we are now ready to make use of Lemma 5.1. If e is chosen to be PL, then T_e and T_e^- are compact polyhedra. Thus the projection maps $q: B \times T_e \rightarrow B$ and $q_-: B \times T_e^- \rightarrow B$ are PL approximate fibrations. We also have a f.p. homotopy equivalence, $\text{id} \times H_-^{-1}H: B \times T_e \rightarrow B \times T_e^-$. In order to apply Lemma 5.1 to this map it should, strictly speaking, be a f.p. sdr. By stabilizing $B \times T_e$ (i.e. multiplying by a large cell) we could easily replace $H_-^{-1}H$ by a sdr, so we will assume that Lemma 5.1 applies to $\text{id} \times H_-^{-1}H$. To use Lemma 5.1 we need to know that $\tau_B = 0$, and by the Sum Theorem it follows that $\tau_B = \chi(B)\tau(H_-^{-1}H)$. We showed in [5] that the image of $\tau(H_-^{-1}H)$ in $\text{Wh}(F \times S^1)$ under the isomorphism $(H_-)_*: \text{Wh}(T_e^-) \rightarrow \text{Wh}(F \times S^1)$ is just $\sigma(F)$, where $\tilde{K}_0(F)$ is identified with a subgroup of $\text{Wh}(F \times S^1)$ in the usual way [11]. Since $\chi(B)\sigma(F) = 0$ we have $\tau_B = 0$.

Now using Lemma 5.1 there is a homeomorphism $h: B \times T_e \times Q \rightarrow B \times T_e^- \times Q$ which is $(q_- \circ \text{proj})^{-1}(\epsilon)$ -homotopic to $\text{id}_B \times H_-^{-1}H \times \text{id}_Q$. This homotopy lifts to a proper homotopy $\text{id}_B \times \tilde{H}_-^{-1}\tilde{H} \times \text{id}_Q \simeq \tilde{h}$, where $\tilde{h}: B \times S_e \times Q \rightarrow B \times S_e^- \times Q$ is a homeomorphism. Choose N large enough so that $\tilde{h}^{-1}(B \times X \times [N, \infty) \times Q)$ lies in $B \times X \times (0, \infty) \times Q$. Define

$$M = (B \times X \times [0, \infty) \times Q) \cap (\tilde{h}^{-1}(B \times X \times (-\infty, N] \times Q)).$$

This is a compact Q -manifold. The triangle

$$\begin{array}{ccc} B \times S_e \times Q & \xrightarrow{\tilde{h}} & B \times S_e^- \times Q \\ \text{proj} \searrow & & \swarrow \text{proj} \\ & B & \end{array}$$

ϵ -commutes, so using mapping cylinder collapses in S_e and S_e^- we observe that the inclusion $M \hookrightarrow B \times S_e \times Q$ is a $\pi^{-1}(\epsilon)$ -equivalence, where $\pi = \text{proj}: B \times S_e \times Q \rightarrow B$. Then the composition

$$M \hookrightarrow B \times S_e \times Q \xrightarrow{\text{id} \times \tilde{H} \times \text{id}} B \times F \times R \times Q \xrightarrow{\text{proj}} B \times F$$

is a $p^{-1}(\epsilon)$ -equivalence, and since M can be triangulated we are done. $\square$

For the proof of Theorem 2 we will need the following version of Theorem 5.2 with dimension estimates.

THEOREM 5.3. *For every integer m there exists an integer n so that if F is a finitely dominated polyhedron of dimension $\leq n$ and B is a compact connected polyhedron of dimension $\leq n$, then the condition $\chi(B) \cdot \sigma(F) = 0$ is necessary and sufficient that for every $\epsilon > 0$ there exists a compact polyhedron K of dimension $\leq m$ and a $p^{-1}(\epsilon)$ -equivalence $K \rightarrow B \times F$.*

PROOF. We can modify the proof of Theorem 5.2, being careful that the dimensions do not grow too large. First we note that since F is finitely dominated and $\dim F \leq n$, we can choose the dimension of the dominating compact polyhedron X to be $\leq n$. This just uses the fact that any map of F into a polyhedron can be homotoped into the n -skeleton. The next change in the proof of Theorem 5.2 comes when the homeomorphism h is constructed. We will replace this with a compact polyhedron Z and CE-PL maps $\alpha: Z \rightarrow B \times T_e$, $\beta: Z \rightarrow B \times T_e^-$ such that β is $(q_-)^{-1}(\epsilon)$ -homotopic to $(\text{id}_B \times H_-^{-1}H)\alpha$. Also we want $\dim Z$ to be bounded in terms of n . Once we have done this we lift α, β to $\tilde{\alpha}: \tilde{Z} \rightarrow B \times S_e$, $\tilde{\beta}: \tilde{Z} \rightarrow B \times S_e^-$, and our desired compact polyhedron K is

$$K = \tilde{\alpha}^{-1}(B \times X \times [0, \infty)) \cap \tilde{\beta}^{-1}(B \times X \times (-\infty, N]),$$

for N a large integer.

To construct Z and the CE-PL maps α, β we need to improve Lemma 5.1. Using the notation of Lemma 5.1 suppose that $\dim E$, $\dim E_1$ and $\dim B$ are all $\leq n$. We want to prove that if $\tau_B = 0$, then for every $\epsilon > 0$ there exists a compact polyhedron Z , $\dim Z \leq m$, and CE-PL maps $\alpha: Z \rightarrow E_1$, $\beta: Z \rightarrow E$ so that β is $p^{-1}(\epsilon)$ -homotopic to $f_1\alpha$. Of course m should depend only on n . Recall that the strategy used in Lemma 5.1 was to use a trick of trading adjacent cancelling torsions so that we were eventually reduced to the case in which the torsion of $f_1|: p_1^{-1}(\Delta) \rightarrow p^{-1}(\Delta)$ is zero, for all simplices in $B^{(r)}$. We first explain how to redo this trick in the context of our desired improvement of Lemma 5.1.

Suppose that $a_1, a_2 \in B$ are close together and we have defined $E_2 = E_1 \cup A_1 \cup A_2$, where there are sdr's $A_i \rightarrow p_1^{-1}(a_i)$ which piece together to yield a f.p. PL sdr $f_2: E_2 \rightarrow E_1$. This gives us a PL approximate fibration $p_2 = p_1 f_2: E_2 \rightarrow B$. Also we assume that the torsions of $f_2|: p_2^{-1}(a_1) \rightarrow p_1^{-1}(a_1)$ and $f_2|: p_2^{-1}(a_2) \rightarrow p_1^{-1}(a_2)$ add up to zero in $\text{Wh}(p^{-1}(*))$. Thus they add up to zero over a small polyhedron P in B which contains a PL arc from a_1 to a_2 . This means that $f_2|: p_2^{-1}(P) \rightarrow p_1^{-1}(P)$ has zero torsion, and so we can find a compact polyhedron W and CE-PL retractions $u: W \rightarrow p_2^{-1}(P)$, $v: W \rightarrow p_1^{-1}(P)$. Define $E_3 = E_2 \cup W$, sewn together along $p_2^{-1}(P)$, and define $f_3: E_3 \rightarrow E_2$ by $f_3|_W = u$ and $f_3 = \text{id}$ otherwise. Now $p_3 = p_2 f_3: E_3 \rightarrow B$ is still an approximate fibration and the torsion of $f_1 f_2 f_3|: p_3^{-1}(a_i) \rightarrow p^{-1}(a_i)$ equals the torsion of $f_1 f_2|: p_2^{-1}(a_i) \rightarrow p^{-1}(a_i)$, for $i = 1, 2$. Also the torsion of $f_1 f_2 f_3|: p_3^{-1}(b) \rightarrow p^{-1}(b)$ equals the torsion of $f_1|: p_1^{-1}(b) \rightarrow p^{-1}(b)$, for all $b \neq a_1, a_2$ in B . So in the context of trading adjacent torsions the space E_3 is as good as the space E_2 . However we now have the advantage that v extends to a CE-PL map $\tilde{v}: E_3 \rightarrow E_1$ for which $\tilde{v}$ is homotopic to $f_2 f_3$ via a homotopy whose image in B is small provided that the diameter of P is small. So if the lemma is true for $p_3: E_3 \rightarrow B$ it is also true for $p_1: E_1 \rightarrow B$.

The above comments tell us that we can now use the torsion trading trick to reduce ourselves to the special case in which the torsion of each $f_1|: p_1^{-1}(\Delta) \rightarrow p^{-1}(\Delta)$ is zero. We will inductively build our desired Z and CE-PL retractions $\alpha: Z \rightarrow E_1$, $\beta: Z \rightarrow E$. For each i let B_i denote the i -skeleton of B . Note that for each vertex v of B_0 , the inclusion $p^{-1}(v) \hookrightarrow p_1^{-1}(v)$ has torsion 0. Thus by expanding each $p_1^{-1}(v)$ we can

form a compact polyhedron Z_0 and CE-PL retractions $\alpha_0: Z_0 \rightarrow E_1$, $\beta_0: (p_1\alpha_0)^{-1}(B_0) \cup E \rightarrow E$ so that $\beta_0((p_1\alpha_0)^{-1}(v)) = p^{-1}(v)$, for each v in B_0 . This completes the first step of the induction. Next note that for each 1-simplex Δ of B_1 , the inclusion $(p\beta_0)^{-1}(\Delta) \hookrightarrow (p_1\alpha_0)^{-1}(\Delta)$ has torsion 0. Thus by expanding each $(p_1\alpha_0)^{-1}(\Delta)$ we can form a compact polyhedron Z_1 and CE-PL retractions

$$\alpha_1: Z_1 \rightarrow Z_0, \quad \beta_1: (p_1\alpha_0\alpha_1)^{-1}(B_1) \cup E \rightarrow (p_1\alpha_0)^{-1}(B_0) \cup E$$

so that β_1 takes each $(p_1\alpha_0\alpha_1)^{-1}(\Delta)$ to $(p_1\alpha_0)^{-1}(\partial\Delta) \cup p^{-1}(\Delta)$. By composing these maps with α_0 and β_0 we retrieve CE-PL retractions

$$\alpha_0\alpha_1: Z_1 \rightarrow E_1, \quad \beta_0\beta_1: (p_1\alpha_0\alpha_1)^{-1}(B_1) \cup E \rightarrow E.$$

This completes the second step of the induction. It should now be clear how this process can be continued to eventually find our desired Z and CE-PL retractions α, β . $\square$

6. Proof of Theorem 2 (infinite-dimensional case). We start by treating the case in which B is a compact connected polyhedron. Choose a finitely dominated polyhedron F , with Wall finiteness obstruction $\sigma(F) \in \tilde{K}_0(F)$, for which $\chi(B) \cdot \sigma(F) = 0$. By Theorem 5.2 there is a compact Q -manifold M and a $\pi^{-1}(\epsilon)$ -equivalence $f: M \rightarrow B \times F$, where $\pi = \text{proj}: B \times F \rightarrow B$ and ϵ is small. Then $\pi f: M \rightarrow B$ is an ϵ' -fibration, for ϵ' small, and so by Theorem 1 of [6] πf is close to an appropriate fibration $p: M \rightarrow B$. Homotopic maps have homotopy fibers which are homotopy equivalent, so p has fiber F .

For the general case in which B is a compact connected ANR we form $B \times Q$ and write $B \times Q = B' \times Q$, where B' is a compact polyhedron. Let F be a finitely dominated polyhedron for which $\chi(B) \cdot \sigma(F) = \chi(B') \cdot \sigma(F) = 0$, and use the above case to get an approximate fibration $p': M \rightarrow B'$ with fiber F . Then $p' \times \text{id}: M \times Q \rightarrow B' \times Q = B$ is also an approximate fibration with fiber F . $\square$

7. Proof of Theorem 2 (finite-dimensional case). In the language of [19] our first step is to choose a PL manifold W^w , $w \geq 5$, which has a single tame end ∞ and an end invariant $\sigma(\infty) \in \tilde{K}_0\pi_1(\infty)$ which is an obstruction to putting a boundary on W . Recall that such a W has the property that π_1 is stable at ∞ and for each cofinite subcomplex K of W , K is finitely dominated and its finiteness obstruction is just $\sigma(\infty)$ (up to sign). By [19, Chapter VIII] all such elements of a given $\tilde{K}_0\pi_1(\infty)$ can be realized in this manner, and if we have $\pi_1(\infty) = Z_{257}$, then we can choose $\sigma(\infty)$ so that $\sigma(\infty) + (-1)^{w-1}\overline{\sigma(\infty)} \neq 0$, where $\bar{\cdot}: \tilde{K}_0\pi_1(\infty) \rightarrow \tilde{K}_0\pi_1(\infty)$ is the duality involution [19, p. 118].

For our next step we will need a construction from [14] which provides us with an open neighborhood $U \subset W$ of ∞ for which there is an approximate fibration $q: U \rightarrow R$ such that $q^{-1}([0, \infty))$ is a closed neighborhood of ∞ . Here is a brief description of how U arises: Using engulfing it is possible to perform the twist-gluing operation of [20] to wrap up the ends of a neighborhood V of ∞ to obtain a compact manifold N^{w+1} and a δ -fibration $u: N \rightarrow S^1$, for δ small. Applying

Theorem 1 of [7] we see that u is close to an approximate fibration $v: N \rightarrow S^1$. The pull-back

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{v}} & R \\ \downarrow & & \downarrow e \\ N & \xrightarrow{v} & S^1 \end{array}$$

provides us with an approximate fibration $\tilde{v}: \tilde{N} \rightarrow R$ for which $\tilde{v}^{-1}((-1, 1))$ is identified with a “sizable” open subset of V . Then engulfing enables us to stretch out this open set to obtain an identification of all $\tilde{N}$ with an open neighborhood U of ∞ which approximately fibers over R as desired. Note that U is finitely dominated and it has two tame ends, ∞_- and $\infty_+ = \infty$. Write $U = K_- \cup K_+$, where K_- and K_+ are closed subcomplexes for which $K_- \subset q^{-1}((-\infty, 1])$ and $K_+ \subset q^{-1}([-1, \infty))$. By the Sum Theorem we have $\sigma(U) = \sigma(K_-) + \sigma(K_+)$, thus $\sigma(U) = \sigma(\infty_-) + \sigma(\infty)$ (up to sign). By the duality formula of [19, p. 119] we have $\sigma(\infty_-) = (-1)^{w-1}\overline{\sigma(\infty)}$. Since we have chosen $\sigma(\infty)$ so that $\sigma(\infty) + (-1)^{w-1}\overline{\sigma(\infty)} \neq 0$ we conclude that U is not homotopy equivalent to a finite complex.

Now consider the product $B^n \times W$, there B^n is our given base manifold of Theorem 2 which has Euler characteristic zero. We are going to carve out of $B \times W$ an approximate fibration $p: M^k \rightarrow B$, where $k = n + w - 1$ and the homotopy fiber of p is U . To do this we apply the Approximate End Theorem of Quinn [18] to the map $\pi = \text{proj}: B \times W \rightarrow B$. In what follows it will be helpful for the reader to be acquainted with §7 of [18] in which the Approximate End Theorem is established. For the time being we will specifically assume that the obstruction which occurs at the bottom of p. 319 of [18] is zero, and show how this yields our desired approximate fibration $p: M \rightarrow B$. Since this obstruction is zero we can carve up $B \times W$ into h -cobordisms where the projection of the homotopies to B are small. More precisely we can find compact clean PL submanifolds N_1, N_2 of $B \times U$ so that:

- (1) $\partial N_1 = \partial_- N_1 \cup \partial_+ N_1$ and $\partial N_2 = \partial_- N_2 \cup \partial_+ N_2$;
- (2) $N_1 \cap N_2 = \partial_+ N_1 = \partial_- N_2$;
- (3) $\partial_- N_1 \subset B \times q^{-1}((-3, -2))$, $\partial_+ N_1 \subset B \times q^{-1}((-1, 1))$, $\partial_+ N_2 \subset B \times q^{-1}((2, 3))$;
- (4) $B \times q^{-1}((-2, -1)) \subset N_1 \subset B \times q^{-1}((-3, 1))$;
- (5) $B \times q^{-1}((1, 2)) \subset N_2 \subset B \times q^{-1}((-1, 3))$;
- (6) there are $\pi^{-1}(\delta)$ -sdr's of N_1 onto $\partial_- N_1$ and onto $\partial_+ N_1$;
- (7) there are $\pi^{-1}(\delta)$ -sdr's of N_2 onto $\partial_- N_2$ and onto $\partial_+ N_2$.

It then follows that there is a $\pi^{-1}(\delta)$ -sdr of $B \times U$ onto $\partial_+ N_1$. Since $\pi: B \times U \rightarrow B$ is a Hurewicz fibration we conclude that $\pi|: \partial_+ N_1 \rightarrow B$ is a δ -fibration whose homotopy fiber is also U . By Theorem 1 of [7], $\pi|: \partial_+ N_1$ is close to an approximate fibration $p: \partial_+ N_1 \rightarrow B$ with homotopy fiber U . This completes the proof of Theorem 2 in case the obstruction vanishes. We now show that this obstruction vanishes so that we can construct our desired “controlled” h -cobordisms.

The first step is to construct a clean neighborhood $N \subset B \times W$ of ∞ for which $(N, \partial N)$ is $(n + w - 3)$ -connected. In the uncontrolled case we just use Siebenmann's thesis [19], but we want to do this with control in B so we use §7 of [18] to construct

N so that $(N, \partial N)$ is $(\delta, n + w - 3)$ -connected for some small δ (i.e. all the homotopies have diameters $< \delta$ in the B -coordinate). We want to carve out of this a clean neighborhood $N_1 \subset N$ of ∞ for which $(N_1, \partial N_1)$ is $(\epsilon, n + w - 2)$ -connected for some small ϵ . Once this is done we can proceed as in [18] to finish the argument. We first choose another $N' \subset N$ which is much closer to ∞ so that $(N', \partial N')$ is also $(\delta, n + w - 3)$ -connected, and then perform a small deformation of the handlebody structure so that

$$N - (N')^\circ = \partial N \cup \{(n + w - 3)\text{-handles}\} \cup \{(n + w - 2)\text{-handles}\}$$

and so that each handle is small. ("Small" generally refers to measurements made in the B -coordinate.) Let A be the union of ∂N and the given $(n + w - 3)$ -handles, and consider the kernel of the boundary map of $Z\pi_1(\infty)$ -modules,

$$\partial: \pi_{n+w-2}(N - (N')^\circ, A) \rightarrow \pi_{n+w-3}(A, \partial N).$$

Recall from [18, p. 319] that $\pi_{n+w-2}(N - (N')^\circ, A)$ may be regarded as a geometric $Z\pi_1(\infty)$ -module G and there is a small projection $u: G \rightarrow G$ for which $\text{Im } u = \text{Ker } \partial$. Precisely what is needed is the following: *There is another geometric $Z\pi_1(\infty)$ -module G' and a small deformation $E: G \oplus G' \rightarrow G \oplus G'$ so that $E(\text{Ker } \partial \oplus G'_1)$ is freely generated by a subset of the preferred bases of G and G' , where $G'_1 \subset G'$ is freely generated by a subset of the preferred basis of G' .* This condition can be met by merely assuming that $\text{Ker } \partial$ is stably free with control in B , i.e., there are geometric modules G_1, G_2 and a small isomorphism $G_1 \cong \text{Ker } \partial \oplus G_2$. The deformation E comes from Lemma 5.4 of [19] and it is small provided that the data which is plugged into the proof in [19] is small.

Recall in the uncontrolled case of [19] it is shown that if N has the homotopy type of a finite complex, then $\text{Ker } \partial$ is stably free, i.e., there are geometric modules G_1, G_2 and an isomorphism $G_1 \cong \text{Ker } \partial \oplus G_2$. We claim that the same proof gives us a small isomorphism (and therefore $\text{Ker } \partial$ is stably free with control) provided that we plug in the additional data that there is a compact polyhedron K and a $\pi^{-1}(\delta)$ -equivalence $K \simeq N$, for some small δ . The dimension of K must be bounded in terms of $n + w$, for when one examines the uncontrolled proof of [19] he finds that layers of cells are attached to N to build its dimension up to $\dim K$, and each time a new layer is attached some control is lost in B . So there only remains the problem of obtaining K . If we choose a closed polyhedral neighborhood $P \subset W$ of ∞ for which $B \times P \subset N$, then Theorem 5.3 tells us that there exists a compact polyhedron K' and a $\pi^{-1}(\delta)$ -equivalence $K' \simeq B \times P$. Since $N - (B \times P)^\circ$ is a compact polyhedron, then our desired K is obtained from K' simply by attaching a finite complex.

□

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